فراغات درشليت لدوال ذات عدد لانهائي من المتغيرات

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Dirichlet Spaces of Functions with Infinite Number of Variables

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قائمة المحتويات

المستخلص

تتركز دراستنا في هذه الرسالة على النقاط الثلاث التالية:

١. تكوين مؤثر تفاضلي جزئي ناقصي ذو عدد لانهائي من المتغيرات وإثبات أنه موجود ومعرف تعريفاً جيداً على حاصل الضرب الممتد اللانهائي من فراغات الدوال ذات متغير واحد والقابلة للتكامل التربيعي و إثبات متباينة جاردنج لذلك المؤثر.

٢. تعريف مؤثر شبه تفاضلي على حاصل الضرب الممتد اللانهائي لفراغات هلبرت، رمزه عبارة عن دالة محددة سالبة ذات عدد لانهائي من المتغيرات، وإثبات تحت شروط معينة أن التمديدات المغلقة لهذه المؤثرات ما هي إلا مولدات لأشباه زمر فيللر.

٣. إثبات أن الشكل الثنائي الذي نطاقه فراغ سوبولف ذو عدد لانهائي من المتغيرات يكون شكل درشليت وذلك باستخدام صيغة ليفي- هنشن للدوال المحددة السالبة ذات عدد لانهائي من المتغيرات لإثبات خاصية ماركوف جزئي للشكل الثنائي وبالتالي نحصل على فراغ درشليت.

تتكون الرسالة من أربعة أبواب، يمكن تلخيصها كمايلي:

الباب الأول: **فراغات هلبرت لدوال ذات عدد لانهائي من المتغيرات**

تضمن العديد من المفاهيم والحقائق الهامة التي نحتاجها كتمهيد لبقية أبواب الرسالة. في الفصل الأول عرضنا النظرية العامة للفراغات ذات المعايير الموجبة والسالبة ثم وضحنا كيفية تكوين سلاسل فراغات هلبرت لدوال ذات عدد منتهي من المتغيرات، وتناولنا بصفه خاصة فراغات سوبولف. في الفصل الثاني والثالث قدمنا نظرية حاصل الضرب الممتد لعدد منتهي ثم لعدد لانهائي من فراغات هلبرت على الترتيب.

الباب الثاني: **تحليل فوريير والدوال المحددة السالبة**

كُرس هذا الباب لدراسة التحليل التوافقي على R^n والذي يعتمد على تحليل فوريير كأهم مقوماته. في الفصل الأول درسنا تحويل فوريير على فراغ شفارتز $S(R^n)$ ثم على الفراغين $L_1(R^n)$ و $L_2(R^n)$. في الفصل الثاني ناقشنا تعريف وخواص الدوال المحددة

الموجبة، وذلك بعد دراسة موجزة لقياسات بورل المحدودة. في الفصل الثالث ركزنا على دراسة أشباه زمر الالتفاف المكونة من قياسات بورل المحدودة ودراسة الدوال المحددة السالبة ثم وضحنا علاقة التناظر بينهما. في الفصل الرابع تناولنا صيغة ليفي- هنشن للدوال المحددة السالبة المتصلة.

الباب الثالث: **المؤثرات التفاضلية الناقصية ذات عدد لانهائي من المتغيرات**

تضمن هذا الباب أول النتائج الجديدة في هذه الرسالة. في الفصل الأول درسنا كيفية إنشاء مؤثر تفاضلي ذو عدد لانهائي من المتغيرات بواسطة تعميم حاصل الضرب الممتد لعدد منتهي من المؤثرات إلى حاصل ضرب ممتد لانهائي. في الفصل الثاني قدمنا النظرية الأساسية لوجود حل لمسألة درشليت ثم عرضنا إثبات متباينة جاردنج لمسألة مؤثر تفاضلي ناقصي معرف على مجال محدود من R^n. في الفصل الثالث عرضنا أولى نتائجنا الجديدة وهي بعنوان متباينة جاردنج لمؤثر تفاضلي ناقصي ذو عدد لانهائي من المتغيرات.

الباب الرابع: **المؤثرات شبه التفاضلية لعدد لانهائي من المتغيرات المولدة لأشباه زمر فيللر**

عرضنا فيه باقي نتائجنا في هذه الرسالة. في الفصل الأول قدمنا النظرية العامة لأشباه الزمر. ثم تناولنا بصفه خاصة أشباه زمر فيللر ومولداتها. في الفصل الثاني درسنا تعريف وخواص الدوال المحددة السالبة ذات عدد لانهائي من المتغيرات وأيضاً صيغة ليفي- هنشن لتلك الدوال. في الفصل الثالث عرضنا ثاني النتائج الجديدة وهي بعنوان المؤثرات شبه التفاضلية ذات عدد لانهائي من المتغيرات والمولدة لأشباه زمر فيللر. في الفصل الرابع قدمنا ما تبقى من النتائج الجديدة حيث أثبتنا أن فراغ هلبرت $W^{\psi^2,1/2}(R^\infty)$ يكون فراغ درشليت.

وقد تم تزويد الرسالة بقائمة للمراجع المستخدمة، إضافةً إلى قائمة بأهم الرموز و المصطلحات المستخدمة و ترجمتها، و ملخص باللغة الإنجليزية.

Abstract

our study in this thesis is concentrated in the following points:

1. Formulate the elliptic differential operator with infinite number of variables and investigate it is well defined on infinite tensor product of spaces of square integrable functions. Under suitable conditions, we prove Garding's inequality for this operator.
2. In the infinite tensor product of Hilbert spaces, we define pseudo differential operators with symbol in terms of negative definite functions. The main result is to show that under suitable conditions closed extensions of these operators form generators of Feller semigroups.
3. Define a Dirichlet form B constructed by a pseudo differential operator with infinite number of variables and prove that B is a sub-Markovine by using of the Levy-Khinchin formula for continuous negative definite functions, which implies that $\left(\mathcal{D}(B), B\right)$ is a Dirichlet space where $\mathcal{D}(B) = W^{\psi^2,1/2}(R^\infty)$.

A brief outline of the contents of thesis is as follows:

Chapter I: **Hilbert Spaces of Functions of Infinitely Many Variables**

This chapter is introductory chapter, which presents the theory of functions of finitely and infinitely many variables. In section one, we study the general theory of spaces with positive and negative norms, then we explain how to construct a chain of Hilbert spaces of functions with finite number of variables. The theory of finite and infinite tensor products of Hilbert spaces and riggings of them is presented in section tow and three.

Chapter II: **Fourier Analysis and Negative Definite Functions**

This chapter is an introduction to harmonic analysis on R^n. In section one, we study the Fourier transform on the Schwarts space $S(R^n)$

then on $L_1(R^n)$ and $L_2(R^n)$. In section tow, we discuss definition and properties of positive definite functions. In section three, we study convolution Semigroups of bounded Borel measures and continuous negative definite functions. Continuous negative definite functions are characterized by the Levy-Khinchin formula and this result is established in section four.

Chapter III: **Elliptic Differential Operators with Infinite Number of Variables**

In section one, we illustrate the constructions of operators with infinite number of variables by generalization the tensor product of finite number of operators to the case of an infinite number of operators. In section tow, we present Dirichlet problem and give a proof of Garding's inequality for linear differential operators defined on bounded open set of R^n. Section three contains our first new result " Garding's inequality for elliptic differential operator with infinite number of variables".

Chapter IV: **Pseudo Differential Operators with Infinite number of Variables Generating Feller Semigroups**

In section one, we give an introduction to the theory of Semigroups and then we concentrate on Feller Semigroups and their generators. In section tow, we study the definition and properties of negative definite functions with infinite number of variables and give the Levy- Khinchin formula in that case. In section three, we present our second new result " Pseudo differential operators with infinite number of variables generating Feller Semigroups". In the last section, we prove that the space $W^{\psi^2,1/2}(R^\infty)$ is Dirichlet space.

(لاتوجد نتائج وتوصيات)

SUMMARY

Interest in the analysis of functions of infinitely many variables and their spaces has increased considerably in recent years in the study of differential equations and boundary value problems [4,62].

An important example of boundary value problems is Dirichlet problem. We mean the problem of finding functions satisfying a partial differential equation defined on a certain domain and satisfying a certain condition on the boundary. Dirichlet problem formulated as follows: Let $\Omega \subset R^n$ be an open set with boundary Γ, $f \in C(\Omega)$ and $g \in C(\Gamma)$. Find all functions u such that

$$Lu = f \text{ in } \Omega \qquad \text{and} \qquad u = g \text{ on } \Gamma$$

hold, where L is a differential operator.

A differential operator L is in the divergence form if it is defined by

$$Lu = \sum_{|\alpha|,|\beta| \leq m} (-1)^{|\beta|} D^{\beta}\left(a_{\alpha\beta}(x) D^{\alpha} u\right)$$

We say that L is strongly elliptic, if there exists a constant $c > 0$ such that

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x) \xi^{\alpha} \xi^{\beta} \geq c|\xi|^{2m}, \qquad \forall x \in \Omega, \quad \forall \xi \in R^n$$

where $a_{\alpha\beta}$ are measurable, bounded functions. An example of such an operator L is the Laplace operator $L = -\Delta$, in this case we get the classical Dirichlet problem where $u \in C^2(\Omega) \cap C(\overline{\Omega})$. When $u = 0$ on Γ we say that a Dirichlet problem is homogeneous [36].

Dirichlet problem arises from variational principles in physics, i.e. from minimization problems of certain functional which have a physical meaning (the so-called energy functional). The Lax-Milgram theorem represents a convenient tool which can be applied to the study of many linear elliptic boundary value problems. It generalizes the Riesz representation theorem for functional defined on Hilbert spaces.

Our main purpose of this thesis is to define a bilinear form B generated by pseudo differential operator with infinite number of variables, defined on a certain anisotropic Sobolev space, and obtain suitable assumptions to construct a Dirichlet form.

By definition a Dirichlet form on $L_2(R^n)$ is a closed symmetric non-negative bilinear form B with the domain $\mathcal{D}(B)$ such that $u \in \mathcal{D}(B)$ implies that

$$v = (u \vee 0) \wedge 1 \in \mathcal{D}(B) \qquad \text{and}$$

$$B(v, v) \leq B(u, u) \;\; \forall\, u \in \mathcal{D}(B)$$

The last property will be called the sub-Markovain property of B. The pair $\big(\mathcal{D}(B), B\big)$ is called Dirichlet space.

The theory of Dirichlet forms on finite dimensional spaces is based on the interplay between analysis (calculus of variations, boundary value problems, potential theory) and probability theory (Brownian motion, stochastic processes, martingale theory) [3]. The extension of the theory to infinite dimensional spaces has been discussed extensively in recent years [3,5,11].

The notion of Dirichlet form had been introduced by A. Beurling and J. Deny [14,28,52] in order to generalize Hilbert spaces methods used in classical potential theory to more general situations. It was first formulated of Dirichlet forms in the finite case by Fukushima [28] and in the infinite case, by Albeverio and Hoegh-Krohn [5]. The strong connection between analysis and probability developed in the late 40's and the 50's when applied the semigroup theory in the study of partial differential equations (Dynkin, Feller, Hunt).

Feller Semigroups play an important role in the theory of Dirichlet forms. It was Fukushima [27,28] who pointed out that Dirichlet forms are in a one to one correspondence to Markove processes, also he proved a one to one correspondence between Dirichlet forms and Feller Semigroups.

To any Dirichlet form one can associate a self adjoint operator, its generator [3,6]. All properties of the form must be reflected in properties of this generator. If the form is local, this generator is a closed extension of a differential operator and it is rather common to use this differential operator to get information about the Dirichlet form. However, in the case of a non local form, this procedure is not very often applicable when the form is given by its general representation depending on time [41], and this out of our study.

One of the most important tool to study Dirichlet forms and its generators is the Fourier transform. It turns out that a class of generators of Dirichlet forms is a class of pseudo differential operators $L(x, D)$, the symbol $L(x, \xi)$ of which is for fixed $x \in R^n$ a continuous negative definite functions with respect to ξ.

In order to solve the Dirichlet problem for a differential operator by using Hilbert space methods (sometimes called the direct method in the calculus of variations), Garding's inequality represents an essential tool [44,67]. For strongly elliptic differential operators, Garding's inequality was proved by L. Garding [31] and its converse by S. Agmon [1]. One can find a proof for Garding's inequality and its converse in F. Stummel [66] for strongly semielliptic operators. More recent results on this subject can be found in [37,47] for a class of differential operators in generalized divergence form.

our new results are concentrated in chapters three and four, which can be summarized in the following three points:

1) Formulate the elliptic differential operator with infinite number of variables and investigate it is well defined on infinite tensor product of spaces of square integrable functions. Under suitable conditions, we prove Garding's inequality for this operator.
2) In the infinite tensor product of Hilbert spaces, we define pseudo differential operators with symbol in terms of negative definite functions. The main result is to show that under suitable conditions closed extensions of these operators form generators of Feller semigroups.
3) Under suitable conditions it is possible to define a Dirichlet form B constructed by a pseudo differential operator with infinite number of variables and prove that B is a sub-Markovine by using of the Levy-Khinchin formula for continuous negative definite functions, which implies that $\big(\mathcal{D}(B), B\big)$ is a Dirichlet space of functions with infinite number of variables where $\mathcal{D}(B) = W^{\psi^2,1/2}(R^\infty)$.

A brief outline of the contents of thesis is as follows:

Chapter I: **Hilbert Spaces of Functions of Infinitely Many Variables**

In this chapter we study the theory of functions of finitely and infinitely many variables. In section one, we study the general theory of spaces with positive and negative norms. The theory of finite and infinite tensor products of Hilbert spaces and riggings of them is presented in section tow and three, with emphasis on Sobolev spaces.

Chapter II: **Fourier Analysis and Negative Definite Functions**

This chapter is an introduction to harmonic analysis on R^n. In section one, we study the Fourier transform on the Schwarts space $S(R^n)$, then on $L_1(R^n)$ and $L_2(R^n)$, besides proving its simplest properties. In section tow, we discuss definition and properties of positive definite functions. One of the central results is Bochner's theorem establishing a one to one correspondence between bounded Borel measures and bounded continuous positive definite functions. In section three, we study convolution Semigroups of bounded Borel measures and continuous negative definite functions, and we discuss the relation between them. Continuous negative definite functions are characterized by the Levy-Khinchin formula and this result is established in section four.

Chapter III: **Elliptic Differential Operators with Infinite Number of Variables**

In section one, we illustrate the constructions of operators with infinite number of variables by generalization the tensor product of a finite number of operators to the case of an infinite number of operators. In section tow, we present the theory of Dirichlet problem and give a proof of Garding's inequality for linear differential operators defined on bounded open set of R^n. Section three contains our first new result " Garding's inequality for elliptic differential operator with infinite number of variables".

Chapter IV: **Pseudo Differential Operators with Infinite number of Variables Generating Feller Semigroups**

In section one, we give an introduction to the theory of Semigroups and then we concentrate on Feller Semigroups and their generators. In section tow, we study the definition and properties of negative definite functions with infinite number of variables and give the Levy- Khinchin formula in that case. In section three, we present our second new result "Pseudo differential operators with infinite number of variables generating Feller Semigroups". In the last section, we prove that the space $\left(\mathcal{D}(B), B\right)$ is Dirichlet space, where $\mathcal{D}(B) = W^{\psi^2,1/2}(R^\infty)$,.